AD-A038 950 WISCONSIN UNIV MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
ABSTRACT LINEAR AND NONLINEAR VOLTERRA EQUATIONS PRESERVING POS--ETC(U)
FEB 77 P CLEMENT, J A NOHEL DAAG29-75-C-0024
UNCLASSIFIED MRC-TSR-1716 NL

1 OF 1
AD
A038950

END
DATE
FILMED
5-77

AD A038950

MRC Technical Summary Report # 1716

ABSTRACT LINEAR AND NONLINEAR VOLTERRA EQUATIONS PRESERVING POSITIVITY

Ph. Clément and J. A. Nohel

12

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

See 1473

February 1977

(Received January 11, 1977)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C.
20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

ABSTRACT LINEAR AND NONLINEAR VOLTERRA EQUATIONS PRESERVING POSITIVITY

Ph. Clément[1,4] and J. A. Nohel[1,2,3]

Technical Summary Report #1716
February 1977

ABSTRACT

Let X be a real or complex Banach space. We study the Volterra equation

$$(v) \qquad u(t) + \int_0^t a(t-s)Au(s)ds = f(t) \quad (0 \leq t \leq T,\ T > 0),$$

where a is a given kernel, A is a bounded or unbounded linear operator from X to X, and f is a given function with values in X (of particular importance is the case $f = u_0 + a * g$, $u_0 \in X$, $g \in L^1(0,T;X)$, $*$ denotes the convolution). We establish sufficient conditions involving a, A, f which insure that solutions of (v) are positive by using certain representation formulas for solutions of (v). We also discuss the positivity of solutions of (v) when A is a nonlinear (m-accretive) operator and we discuss several examples when A is a partial differential operator.

AMS (MOS) Subject Classifications: 45D05, 45N05, 45G99, 45M99, 47H05, 47H10, 47H15

Key Words: Volterra integral equations, abstract integral equations, unbounded operators, m-accretive operators, semigroups, logarithmic convexity, complete monotonicity, resolvent kernel, fundamental solution, positivity of solutions, Yosida approximation

Work Unit Number 1 (Applied Analysis)

Sponsored by:

1) The United States Army under Contract No. DAAG29-75-C-0024;
2) The United States Army under Grant Number DAAG29-77-G-0004;
3) The National Science Foundation under Grant Number MCS 75-21868.
4) Fonds National Suisse.

ABSTRACT LINEAR AND NONLINEAR VOLTERRA EQUATIONS PRESERVING POSITIVITY

Ph. Clément[1,4] and J. A. Nohel[1,2,3]

1. Introduction and Principal Results.

Let X be a real or complex Banach space. We study the linear Volterra equation

$$u(t) + a * Au(t) = f(t) \qquad (0 \leq t \leq T;\ T > 0) \tag{1.1}$$

where $a * Au(t) = \int_0^t a(t-s)Au(s)ds$, a is a given real kernel, A is a bounded or unbounded linear operator from X to X and f is a given function with values in X.

An important and perhaps the most useful special case of (1.1) for certain applications is the linear equation

$$u(t) + a * Au(t) = u_0 + a * g(t) \qquad (0 \leq t \leq T;\ T > 0) \tag{1.1a}$$

where $u_0 \in X$ and the given function $g \in L^1(0,T;X)$. We will establish conditions on the kernel a and the operator A which insure that the respective solutions operators for (1.1) and (1.1a) preserve a convex cone in X (see Theorems 3 and 4). We then consider in Section 3 a nonlinear problem of the form (1.1) in which A is a m-accretive operator. Finally, in Section 4 we discuss three examples to illustrate the theory. Example 3 was proposed to us by Professor L. A. Peletier. We are grateful to Professor M. G. Crandall for discussing Example 3 with us.

We will suppose throughout that the following assumptions are satisfied.

(H_1) $A : D(A) \subseteq X \to X$ and $-A$ generates a linear continuous contraction semi-group on X, which we shall denote by $e^{-\omega A}$ $(\omega \geq 0)$.

(H_2) $a \in L^1(0,T;\mathbb{R})$

(H_3) $f \in L^1(0,T;X)$

Sponsored by:

1) The United States Army under Contract No. DAAG29-75-C-0024;
2) The United States Army under Grant Number DAAG29-77-G-0004;
3) The National Science Foundation under Grant Number MCS 75-21868.
4) Fonds National Suisse.

Definition 1. We say that $u : [0,T] \to X$ is a strong solution of (1.1) if $u \in L^1(0,T;X)$, $u(t) \in D(A)$ a.e. on $[0,T]$, $Au \in L^1(0,T;X)$, and u satisfies (1.1) a.e. on $[0,T]$.

We denote the norm in X by $\|\cdot\|$. If B is a linear unbounded operator on X, we use the notation $X_B = D(B)$; if $u \in X_B$, its graph norm is denoted by $\|u\|_{X_B} = \|u\| + \|Bu\|$. Of particular interest are the spaces X_A and X_{A^2} where A satisfies (H_1). Recall that the space X_A is dense in X and X_{A^2} is dense in X_A; see [15, Theorem 2.9, pg. 8]. If u is a strong solution of (1.1), Definition 1 states that $u \in L^1(0,T;X_A)$.

To discuss solutions of (1.1) and (1.1a) we make use of the operators R and S defined respectively by the equations

$$\text{(R)} \qquad u(t) + a * Au(t) = a(t)x \quad (x \in X_A;\ 0 \le t \le T)$$

$$\text{(S)} \qquad u(t) + a * Au(t) = x \quad (x \in X_A;\ 0 \le t \le T)\ .$$

It follows that under the assumptions of Theorem 1 below, equations (R) and (S) each have a unique strong solution which we write respectively as $R(t)x$ and $S(t)x$. While the operators R and S are so defined for $x \in X_A$, Theorem 1 together with a density argument shows that R and S can be extended uniquely as bounded operators in $L^1(0,T;X)$ and $C(0,T;X)$ respectively.

Our main result for the linear case is

Theorem 1. Let (H_1), (H_2) be satisfied.

(i) Let the kernel a satisfy the following condition

(H_4) For every $\lambda \ge 0$, the unique solution $r(t,\lambda) \in L^1(0,T;\mathbb{R})$ of the scalar equation (resolvent equation)

$$r(t) + \lambda a * r(t) = a(t) \qquad (0 \le t \le T)$$

satisfies $r(t,\lambda) \ge 0$ a.e. on $[0,T]$.

Then for every $x \in X_A$, the equation (R) has a unique strong solution which we denote by $R(t)x$, $0 \le t \le T$. Moreover, for almost every $t \in [0,T]$, there exists a positive measure μ_t on $\mathbb{R}^+$, depending only on the kernel a, such that

$$
(1.2) \quad \begin{cases} R(t)x = \int_0^\infty e^{-\omega A} x \, d\mu_t(\omega) \\ \\ a(t) = \int_0^\infty d\mu_t(\omega) \end{cases} \qquad t \in [0,T] \quad \text{a.e.}
$$

and the following estimates are satisfied:

$$
(1.3) \qquad \|Rx\|_{L^1[0,T;Y]} \le \|a\|_{L^1[0,T;\mathbb{R}]} \|x\|_Y ,
$$

where $Y = X$ or X_A or X_{A^n} and

$$
(1.4) \qquad \|R * v\|_{L^p[0,T;Y]} \le \|a\|_{L^1[0,T;\mathbb{R}]} \|v\|_{L^p[0,T;Y]} \quad (1 \le p \le \infty) .
$$

(ii) Let the kernel a satisfy the assumptions (H_4) and:

(H_5) $\left\{\begin{array}{l} \text{For every } \lambda \ge 0, \text{ the unique solution } s(t,\lambda) \text{ (absolutely continuous on } [0,T]) \\ \text{of the scalar equation} \\ s(t) + \lambda a * s(t) = 1 \quad (0 \le t \le T) \\ \textit{satisfies } s(t,\lambda) \ge 0, \ 0 \le t \le T. \end{array}\right.$

Then for every $x \in X_A$ the equation (S) has a unique strong solution which we denote by $S(t)x$, $0 \le t \le T$. Moreover, for every $t \in [0,T]$, there exists a probability measure ν_t on $\mathbb{R}^+$ depending only on the kernel a, such that

$$
(1.5) \qquad S(t)x = \int_0^\infty e^{-\omega A} x \, d\nu_t(\omega) \quad (t \in [0,T]) ,
$$

and the following estimates hold:

$$
(1.6) \qquad \|S(t)x\|_Y \le \|x\|_Y ,
$$

$$
(1.7) \qquad \|S * v\|_{C[0,T;Y]} \le \|v\|_{L^1[0,T;Y]} ,
$$

where $Y = X$ or X_A or X_{A^n}.

Remark 1.1. If $a \equiv 1$, then $R(t) = S(t) = e^{-tA}$ and $\mu_t = \nu_t$ = the Dirac measure at t.

Assumptions (H_4) and (H_5) require some clarification.

Proposition 1. (i) Let (H_2) be satisfied and let $a \in C(0,T)$ and $a(t) > 0$. If $\log a(t)$ is *convex on* $(0,T)$ *then* (H_4) *is satisfied on* $[0,T]$.

(ii) *Let* (H_2) *be satisfied and let* $a(t)$ *be nonnegative and nonincreasing on* $(0,T)$. *Then* (H_5) *is satisfied on* $[0,T]$.

While the content of Proposition 1 is implicitly contained in the literature (see [7], [8], [12] and [14]), we give the proof in Appendix 1. In the literature the results are for t on the infinite interval and under slightly stronger assumptions.

Remark 1.2. It is useful to observe that

$$s(t,\lambda) = 1 - \lambda \int_0^t r(\sigma,\lambda)d\sigma$$

where r and s are defined in (H_4) and (H_5) respectively. This follows from the fact that $a * s = 1 * r$, together with the equation defining s. Thus if (H_4) and (H_5) are satisfied on $[0,T]$ for every $T > 0$ then $\int_0^T r(t,\lambda)dt \le \int_0^T a(t)dt$ and $0 \le \int_0^\infty r(t,\lambda)dt \le \frac{1}{\lambda}$, $\lambda > 0$; in particular, $r(t,\lambda) \in L^1(0,\infty)$, $\lambda > 0$.

Remark 1.3. If $a(t)$ satisfies (H_2) and is completely monotonic on $(0,T)$, then a satisfies (H_4) and (H_5), see [7], [14].

Remark 1.4. We also note that, if $a(t) = e^t$, then (H_4) is satisfied but not (H_5). However, (H_5) does not imply (H_4). To see this, take $a(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$. Then by Proposition 1 (ii), (H_5) is satisfied. But for $\lambda = 1$, as shown by Levin [12; example following Theorem 1.4], $r(1,t) < 0$ for some $1 < t < 2$.

Theorem 1 is used to deduce the following results about solutions of equations (1.1) and (1.1a).

Theorem 2. (i) *Let the assumptions* (H_1), (H_2), (H_4) *and* $g \in L^1(0,T;X_A)$ *be satisfied. Then the equation*

$$u(t) + a * Au(t) = a * g(t) \qquad (0 \le t \le T) \tag{1.8}$$

has a unique strong solution u *given by*

$$u = R * g, \tag{1.9}$$

where R *is the solution of equation* (R) *given by* (1.2), *and* (*by* (1.3))

$$\|u\|_{L^1(0,T;X)} \le \|a\|_{L^1(0,T;\mathbb{R})} \|g\|_{L^1(0,T;X)}. \tag{1.10}$$

(ii) Let the assumptions (H_1), (H_2), (H_4), (H_5) and

$$(H_6)\quad \begin{cases} f = f_1 + f_2 \ \text{ where } \ f_1 \in L^p(0,T;X_{A^2}) \ \ 1 \le p \le \infty \\ \text{and } \ f_2 \in W^{1,1}(0,T;X_A), \ \text{ where } \ W^{1,1} \ \text{ is the usual Sobolev space,} \end{cases}$$

be satisfied. Then equation (1.1) has a unique strong solution $u = u_1 + u_2$ where

(1.11) $$u_1(t) = f_1(t) - R * Af_1(t) \quad \text{a.e. on } [0,T] ,$$

and

(1.12) $$u_2(t) = S(t)f_2(0) + S * f_2'(t) \qquad t \in [0,T] ,$$

where S is the solution of equation (S) given by (1.5); moreover there is a constant $c = c(T) > 0$ such that

(1.13) $$\|u\|_{L^1(0,T;X)} \le c\{ \|f_1\|_{L^1(0,T;X_A)} + \|f_2\|_{W^{1,1}(0,T;X)} \} .$$

Remark 2.1. If A is any bounded linear operator, then $X = X_A = X_{A^2}$ and the existence and uniqueness of solutions of (1.1), with only $a \in L^1(0,T;\mathbb{R})$, $f \in L^1(0,T;X)$ is well-known. In the case when A is not bounded, existence and uniqueness results for solutions of (1.1) have been obtained by Friedman and Shinbrot [9], even for the case $A(t)$ where $A(t)$ generates an analytic semi-group under different conditions both for the kernel and the function f with, however, different objectives than ours.

Remark 2.2. Formula (1.11) is well-known when A is a bounded operator; formula (1.12) has also been employed in [8], [9] where S is called a fundamental solution of (1.1).

Remark 2.3. In the unbounded case we may define a weak solution of (1.1) as follows: there exist sequences $\{u_n\}$, $\{f_n\}$ where each $f_n \in L^1(0,T;X)$ and each u_n is a strong solution of (1.1) with $f = f_n$ such that $f_n \to f$ and $u_n \to u$ in $L^1(0,T;X)$. From (1.13) it follows that if $f \in L^1(0,T;X_A) + W^{1,1}(0,T;X)$, then equation (1.1) possesses a unique weak solution. (Note that $L^1(0,T;X_{A^2})$ is dense in $L^1(0,T;X_A)$ with respect to the norm in $L^1(0,T;X)$; similarly for $W^{1,1}(0,T;X_A)$ in $W^{1,1}(0,T;X)$. A similar remark applies to (1.8).

Remark 2.4. If $f_1 = 0$, then conclusion (1.13) can be strengthened to:

(1.14) $$\|u\|_{C(0,T;X)} \le c\|f_2\|_{W^{1,1}(0,T;X)} .$$

Remark 2.5. Since the kernel is real, the case when X is a real Banach space can be treated as a special case of the complex case: If $\tilde{X} = X + iX$, the operator $\tilde{A}(x + iy) := Ax + iAy$ satisfies (H_1) whenever A satisfies (H_1). Therefore, we can restrict ourselves to the complex case.

Remark 2.6. If $a(t) = \delta(t)$ where $\delta(t)$ is the Dirac measure, then (1.1) reduces to $u(t) + Au(t) = f(t)$, and

$$S(t) = (I + A)^{-1} = \int_0^\infty e^{-\omega A} e^{-\omega} d\omega \quad [19;\ p.\ 240] . \tag{1.15}$$

The kernel $a(t) = \delta(t)$ does not satisfy (H_2). However, $\delta(t)$ can be approximated by kernels $a_\sigma(t) = \frac{1}{\sigma} e^{-\frac{t}{\sigma}}$ $(\sigma \to 0^+)$; each a_σ satisfies (H_2), (H_4), (H_5) so that $a(t) = \delta(t)$ is a limiting case of our theory and the corresponding measures $\nu_t^{(\sigma)}$ approach the measures ν_t in (1.15) of density $e^{-\omega}$, independent of t, as $\sigma \to 0^+$.

By (1.2) and (1.5), $R(t)$ and $S(t)$ are respectively positive and convex "combinations" of contraction semigroups $e^{-\omega A}$. From this observation we obtain the following applications of Theorems 1 and 2 which we state as Theorems 3 and 4.

Theorem 3. *Let (H_1), (H_2), (H_4) be satisfied. Let P be a closed convex cone in X, such that*

$$(I + \lambda A)^{-1} P \subseteq P \quad \textit{for every} \quad \lambda \geq 0 . \tag{1.16}$$

Then

$$R(t)P \subseteq P \quad \text{a.e. on} \quad [0,T] . \tag{1.17}$$

Moreover, if in equation (1.8) $g(t) \in P$ a.e., then the solution u of (1.8) lies in P a.e. on $[0,T]$. If in (1.1) $f \in L^1(0,T;X_A)$ and $Af(t) \in P$ a.e. on $[0,T]$, then

$$f(t) \in u(t) + P \quad \text{a.e. on} \quad [0,T] , \tag{1.18}$$

where u is the (weak) solution of (1.1); in particular, if P is a positive cone in X, the last statement is equivalent to the "maximum principle":

$$u(t) \leq f(t) \quad \text{a.e. on} \quad [0,T] . \tag{1.19}$$

The proof of (1.17) in Theorem 3 is an immediate consequence of formula (1.2) for the operator R, together with the standard fact that assumption (1.16) implies that $e^{-\omega A}$ maps P into P for every $\omega \in \mathbb{R}^+$. Having established (1.17), the remaining conclusions of Theorem 3 follow from the representation formula (1.11).

Remark 3.1. If one studies equation (1.8) in the scalar case, one takes $A = \lambda \geq 0$ to satisfy (H_1). If (H_2) is satisfied and if $P = \mathbb{R}^+$, then the condition (H_4) is necessary and sufficient in order to guarantee that the solution u of (1.8) satisfies $u(t) \geq 0$ for every $g \geq 0$. Thus one cannot hope to improve on condition (H_4) in the abstract case.

Theorem 4. Let (H_1), (H_2), (H_4), (H_5) be satisfied. Let P be a closed convex cone in X satisfying (1.16). Then

$$S(t)P \subseteq P \quad \text{for} \quad 0 \leq t \leq T . \tag{1.20}$$

(i) Moreover, if $u_0 \in P$ and if $g(t) \in P$ a.e. in equation (1.1a), then the solution u of (1.1a) lies in P for almost every $t \in [0,T]$.

(ii) If in (1.1), $f \in W^{1,1}[0,T;X]$ where $f(0) \in P$ and $f'(t) \in P$ a.e. on $[0,T]$, then the (weak) solution u of (1.1) lies in P for every $t \in [0,T]$. (The last assertion holds for any closed convex set P in X).

(iii) Moreover, if X is a real Hilbert space, and if the function $\varphi : X \to [0,\infty]$ is convex, lower semicontinuous, proper and satisfies

$$\varphi((I + \lambda A)^{-1} x) \leq \varphi(x) \quad \text{for every} \quad \lambda \geq 0 \quad \text{and every} \quad x \in X , \tag{1.21}$$

then

$$\varphi(S(t)x) \leq \varphi(x) \quad \text{for every} \quad t \in [0,T] \quad \text{and every} \quad x \in X . \tag{1.22}$$

The proof of (1.20) in Theorem 4 follows from formula (1.5) for the operator S, together with the observation that assumption (1.16) implies that $e^{-\omega A}$ maps P into P for every $\omega \in \mathbb{R}^+$. Then conclusion (i) of Theorem 4 follows from (1.9), (1.12) with $f(t) \equiv u_0$, and the fact that the operators R and S each map P into P. Similarly, conclusion (ii) follows from (1.12). To establish (iii) recall that assumption (1.21) implies that

$$\varphi_\lambda(e^{-\omega A}x) \leq \varphi_\lambda(x) \quad \text{for every} \quad \omega \geq 0, \ \lambda > 0, \ x \in X ,$$

where φ_λ is the Yosida approximation of φ, [3, Proposition 2.11]. Then (1.22) follows from (1.5), Jensen's inequality and $\sup_{\lambda>0} \varphi_\lambda(x) = \varphi(x)$, [3, Proposition 2.11].

Remark 4.1. Conclusion (ii) of Theorem 4 is an abstraction of a result of Levin [12; Lemma 1.3] in $\mathbb{R}^1$. His result is

> "Let $a \in L^1_{loc}(0,\infty)$, $a(t)$ nonnegative nonincreasing on $(0,\infty)$. Let $f \in C[0,\infty)$ be nonnegative and nondecreasing on $[0,\infty)$. Then the solution x of the equation
>
> $$x(t) + a * x(t) = f(t) \qquad (0 \leq t < \infty)$$
>
> satisfies $0 \leq x(t) \leq f(t)$".

This result is also an immediate consequence of Proposition 1 (ii) and of the formula

$$x(t) = S(t)f(0) + \int_0^t S(t-\sigma)df(\sigma) .$$

Levin's proof in [12] is different; he improves his result by a smoothing argument which permits him to remove the assumption $f \in C[0,\infty)$. This is also evident from the preceding formula.

In Theorem 4 (ii) both assumptions (H_4) and (H_5) are used. It is of interest to note that in the abstract case the assumption (H_5) (which is satisfied when a is positive and nonincreasing) is not sufficient to insure that S maps P into P when condition (1.16) is satisfied. To see this we consider the following example in $\mathbb{R}^2$.

Let

$$a(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t \geq 1 , \end{cases} \tag{1.23}$$

and consider for $\alpha > 0$ the operator A_α defined by

$$A_\alpha = U^T \Lambda_\alpha U \quad \text{where} \tag{1.24}$$

$$\Lambda_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1+\alpha \end{pmatrix}, \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

For every $\alpha > 0$, the real matrix A_α is symmetric and positive definite. Thus $-A_\alpha$ generates a contraction semigroup on $\mathbb{R}^2$, with the usual Euclidean norm. If P is the cone $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$, then it is easily checked that $(I + \lambda A_\alpha)^{-1} P \leq P$ for every $\alpha > 0$, $\lambda > 0$, so that (1.16) is satisfied.

Corresponding to the kernel a defined by (1.23), the function $s(t, \lambda)$ of (H_5) is

$$s(t, \lambda) = \begin{cases} e^{-\lambda t} & \text{if } 0 \leq t < 1 \\ e^{-\lambda t} + \lambda(t-1)e^{-\lambda(t-1)} & \text{if } 1 \leq t \leq 2\,, \end{cases} \tag{1.25}$$

and clearly (H_5) is satisfied on the interval $0 \leq t \leq 2$.

We next compute the operator S_α corresponding to A_α. Consider the equation

$$u + a * A_\alpha u = x, \quad x \in \mathbb{R}^2 . \tag{1.26}$$

By setting $v = Uu$, $y = Ux$ equation (1.26) is transformed to the equivalent diagonal form

$$v + a * \Lambda_\alpha v = y\,, \tag{1.27}$$

which by the definition of $s(t, \lambda)$ in (H_5) gives the solution

$$v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} s(t, 1)y_1 \\ s(t, 1+\alpha)y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} s(t, 1)[x_1 + x_2] \\ s(t, 1+\alpha)[-x_1 + x_2] \end{pmatrix} .$$

Thus the solution of (1.26) is

$$u(t) = \frac{1}{2} \begin{pmatrix} s(t, 1)[x_1 + x_2] - s(t, 1+\alpha)[-x_1 + x_2] \\ s(t, 1)[x_1 + x_2] + s(t, 1+\alpha)[-x_1 + x_2] \end{pmatrix},$$

and the operator $S_\alpha(t)$ is

$$S_\alpha(t) = \frac{1}{2} \begin{pmatrix} s(t, 1) + s(t, 1+\alpha) & s(t, 1) - s(t, 1+\alpha) \\ s(t, 1) - s(t, 1+\alpha) & s(t, 1) + s(t, 1+\alpha) \end{pmatrix} .$$

To show that (H_5) is not sufficient to prove that S_α maps $P = R_2^+$ into P, it is sufficient to have $s(t, 1) - s(t, 1+\alpha) < 0$ for some $t > 0$ and for some $\alpha > 0$. Observe that from (1.25)

$$\frac{\partial s}{\partial \lambda}(t, \lambda) = -e^{-\lambda(t-1)}[\lambda(t-1)^2 - (t-1) + te^{-\lambda}] \quad \text{for} \tag{1.28}$$

$1 \leq t \leq 2$, $\lambda > 0$. Thus $\frac{\partial s}{\partial \lambda}(1 + \frac{1}{10}, 1) > 0$, so that there exists $\alpha > 0$ such that

$s(1 + \frac{1}{10}, 1) - s(1 + \frac{1}{10}, 1 + \alpha) < 0$, which establishes the claim.

We note the above argument also shows that (H_5) does not imply that $s(t, \lambda)$ is completely monotonic in λ. (See remarks following Lemma 2.1 below).

2. Proof of Theorems 1 and 2.

We will prove Theorems 1 and 2 in two main steps. We first consider the case when A is a bounded operator. In this case, by Remarks 2.1 and 2.2, it suffices to prove the representation formulas (1.2) and (1.5); for, having these one immediately has the estimates (1.3), (1.4), (1.6), (1.7) as well as the conclusions of Theorem 2. We then consider the case when A is an unbounded operator as a limiting situation of the bounded case using the Yosida approximation of A. The case where A is bounded is further divided into two parts:

(i) scalar case. We require the following preliminary result.

Lemma 2.1. *If $a(t)$ satisfies assumptions (H_2), (H_4), then $r(t, \lambda)$, defined in (H_4), is completely monotonic in λ for $0 < \lambda < \infty$ for $t \in [0,T]$ a.e. If moreover $a(t)$ satisfies (H_5), then $s(t,\lambda)$, defined in (H_5) is completely monotonic in λ for $0 \leq \lambda < \infty$ for every $t \in [0,T]$.*

Proof of Lemma 2.1. We consider the equations

$$r(t,\lambda) + \lambda a * r(t,\lambda) = a(t) \tag{2.1}$$

$$s(t,\lambda) + \lambda a * s(t,\lambda) = 1 \tag{2.2}$$

of assumptions (H_4) and (H_5) respectively with λ complex rather than $\lambda \geq 0$. Let E denote the spaces $L^1(0,T;\mathbb{C})$ or $C(0,T;\mathbb{C})$. Define the operator $K : E \to E$ by $K_x(t) = a * x(t)$ $(x \in E)$. K is a bounded linear operator with spectrum $\sigma(K) = 0$. Thus for $u \in E$, the function v defined by $v(\lambda) = (I + \lambda K)^{-1}u$, $\lambda \in \mathbb{C}$, is an entire function of λ with values in E. By differentiation and induction one has the formula:

$$(-1)^n \frac{d^n}{d\lambda^n} v(\lambda) = n!\, K_\lambda^n v(\lambda), \qquad n = 0,1,2,\ldots \tag{2.3}$$

where the operator K_λ is defined by

$$K_\lambda = K(I + \lambda K)^{-1}. \tag{2.4}$$

We claim that

$$(2.5) \qquad K_\lambda x(t) = \int_0^t r(t-s,\lambda)x(s)ds \quad (x \in E) .$$

To prove (2.5) take the convolution product of both sides of (2.1) by $x \in L^1(0,T;\mathbb{C})$, obtaining

$$r(t,\lambda) * x(t) + \lambda a * r(t,\lambda) * x(t) = a * x(t) .$$

Thus $u_\lambda(t) = r(t,\lambda) * x(t)$ satisfies the equation

$$u_\lambda(t) + \lambda a * u_\lambda(t) = a * x(t) ;$$

by uniqueness of solutions of this scalar equation and by the definition of K_λ in (2.4) this shows that $u_\lambda(t) = K_\lambda x(t)$ and proves (2.5).

For $\lambda \geq 0$, assumption (H_4) implies that the operators K_λ map the set of non-negative real functions in E into itself. To prove the first assertion of Lemma 2.1, consider $v_a(\lambda) = (I + \lambda K)^{-1}a$; then $v_a(\lambda)(t) = r(t,\lambda)$ a.e. in $[0,T]$, $r(t,\lambda) \geq 0$ by (H_4), and by (2.3), (2.5) $(-1)^n \frac{\partial^n}{\partial\lambda^n} r(t,\lambda) \geq 0$ a.e. in $[0,T]$ for $0 < \lambda < \infty$. To prove the second assertion of Lemma 2.1, take $v_1(\lambda) = (I + \lambda A)^{-1}1$; then $v(\lambda)(t) = s(t,\lambda) \geq 0$ by (H_5), and complete the proof as above. This completes the proof of Lemma 2.1.

It should be noted that the second assertion of Lemma 2.1 is stated by Friedman [8, lemma 2.7] under only the hypothesis that $a \geq 0$ and nonincreasing. However, his proof also uses (H_4). (He should also require (H_4) for his Theorem 5.2, p. 144). To see that (H_5) is not sufficient for the complete monotonicity of $s(t,\lambda)$ with respect to λ, we consider again the kernel a defined in (1.23). The corresponding function $s(t,\lambda)$ is given by (1.25) and $s(t,\lambda) \geq 0$, for $0 \leq t \leq 2$. However, as seen from (1.28), $\frac{ds}{d\lambda}(1 + \frac{1}{10}, 1) > 0$.

We shall next obtain representations of the entire functions $r(t,\lambda)$, $s(t,\lambda)$ for $\mathrm{Re}\,\lambda \geq 0$.

By Lemma 2.1 and Bernstein's theorem [18], there exists a positive finite measure μ_t on $\mathbb{R}^+$ such that

$$
(2.6) \qquad \begin{cases} r(t,\lambda) = \int_0^\infty e^{-\omega\lambda} d\mu_t(\omega) & (\mathrm{Re}\lambda > 0;\ t \in [0,T] \ \text{a.e.}) \\ a(t) = \int_0^\infty d\mu_t(\omega) & (t \in [0,T] \ \text{a.e.}) . \end{cases}
$$

Similarly, using $s(0,\lambda) = 1$, there exists a probability measure ν_t on $\mathbb{R}^+$ such that

$$
(2.7) \qquad s(t,\lambda) = \int_0^\infty e^{-\omega\lambda} d\nu_t(\omega) \quad (\mathrm{Re}\,\lambda \geq 0;\ t \in [0,T]) .
$$

Thus (2.6) and (2.7) correspond to formulas (1.2) and (1.5).in the scalar case.

(ii) A is a bounded operator satisfying (H_1). By a standard argument equations (R) and (S) possess for every $x \in X$ a unique solution which we denote by $R(t)x$ and $S(t)x$ respectively. We first prove the representation formulas (1.2) and (1.5) for the operators A_ε defined by

$$
(2.8) \qquad A_\varepsilon = \varepsilon I + A \quad (1 > \varepsilon > 0) .
$$

Define the operators R_ε and S_ε by the formulas

$$
(2.9) \qquad R_\varepsilon(t)x = \frac{1}{2i\pi} \int_{C_\varepsilon} r(t,\lambda)(\lambda I - A_\varepsilon)^{-1} x \, d\lambda \quad (0 \leq t \leq T) ,
$$

$$
(2.10) \qquad S_\varepsilon(t)x = \frac{1}{2i\pi} \int_{C_\varepsilon} s(t,\lambda)(\lambda I - A_\varepsilon)^{-1} x \, d\lambda \quad (0 \leq t \leq T) ,
$$

where $x \in X$, $r(t,\lambda)$, $s(t,\lambda)$ are defined by (2.1) and (2.2) respectively for $\lambda \in \mathbb{C}$. C_ε is a closed contour in the complex λ plane, oriented counterclockwise, consisting of a finite number of rectifiable Jordan arcs and such that $C_\varepsilon = \partial U_\varepsilon$, where U_ε is an open set containing the spectrum of A_ε. The integral in (2.9), (2.10) are the usual Dunford integrals [19, p. 225]. It is shown by Friedman [8, Theorem 3.1] that $S_\varepsilon(t)x$ defined by (2.10) is the unique solution of equation (S) with A replaced by A_ε. An entirely analogous argument shows that $R_\varepsilon(t)x$ defined by (2.9) is the unique solution of equation (R) with A replaced by A_ε.

We next observe that the spectrum $\sigma(A_\varepsilon)$ is contained in the half plane $\mathrm{Re}\,\lambda \geq \varepsilon$, and, if $\epsilon < 1$, in the ball of radius $1 + \|A\|$. Thus we may choose C_ε to be the rectangle bounded by the segments joining the points $(\frac{\varepsilon}{2} - i(2 + \|A\|))$, $((2 + \|A\|)(1 - i))$, $((2 + \|A\|)(1 + i))$, $(\frac{\varepsilon}{2} + i(2 + \|A\|))$ oriented counterclockwise. Using the representation (2.6) in (2.9) under assumption (H_4) and the representation (2.7) in (2.10) under assumptions (H_4), (H_5) we obtain

$$R_\varepsilon(t)x = \int_0^\infty e^{-\omega A_\varepsilon} x\, d\mu_t(\omega) \qquad (x \in X)\,, \tag{2.11}$$

$$S_\varepsilon(t)x = \int_0^\infty e^{-\omega A_\varepsilon} x\, d\nu_t(\omega) \qquad (x \in X)\,. \tag{2.12}$$

The proofs of (2.11), (2.12) follow from a theorem on the Dunford integral [19, p. 226], together with Fubini's theorem and the definition of the operator $e^{-\omega A_\varepsilon}$ by

$$e^{-\omega A_\varepsilon} x = \frac{1}{2\pi i} \int_{C_\varepsilon} e^{-\omega\lambda} (\lambda I - A_\varepsilon)^{-1} x\, d\lambda \qquad (x \in X)\,.$$

Thus formulas (2.11), (2.12) establish (1.2) and (1.5) respectively with $A = A_\varepsilon$. We next let $\varepsilon \to 0^+$. We first show that

$$R_\varepsilon(t)x \to z(t) = \int_0^\infty e^{-\omega A} x\, d\mu_t(\omega) \quad \text{in} \quad L^1(0, T; X)\,. \tag{2.13}$$

We then show that $z(t)$ is the unique solution of equation (R). Substituting (2.8) in (2.11) we have

$$\left\| R_\varepsilon(t)x - \int_0^\infty e^{-\omega A} x\, d\mu_t(\omega) \right\| = \left\| \int_0^\infty (e^{-\varepsilon\omega} - 1) e^{-\omega A} d\mu_t(\omega) \right\|\,.$$

Therefore, by a simple application of Lebesgue's dominated convergence theorem

$$\lim_{\varepsilon \to 0^+} \left\| R_\varepsilon(t)x - \int_0^\infty e^{-\omega A} x\, d\mu_t(\omega) \right\| = 0 \quad \text{a.e. on} \quad [0, T]\,.$$

Moreover, since $e^{-\omega A}$ is a contraction semigroup, we have

$$\|R_\varepsilon(t)x\| \le \int_0^\infty \|e^{-\varepsilon\omega}e^{-\omega A}x\| d\mu_t(\omega) \le \|x\| a(t) \quad \text{a.e.} \tag{2.14}$$

Since $a \in L^1(0,T)$, another application of Lebesgue's theorem establishes (2.13).

We next show that the function z defined in (2.13) is the unique solution of equation (R). We know that $R_\varepsilon(t)x$ is the unique solution of the equation

$$(R_\varepsilon) \qquad u_\varepsilon(t) + a * Au_\varepsilon(t) + \varepsilon a * u_\varepsilon(t) = a(t) \quad \text{a.e.}$$

Observe that by (2.14)

$$\|u_\varepsilon\|_{L^1(0,T;X)} \le \|x\| \int_0^T a(t)dt .$$

Consequently $\varepsilon a * u_\varepsilon \to 0$ in $L^1(0,T;X)$ as $\varepsilon \to 0^+$. Since $u_\varepsilon \to z$ in $L_1(0,T;X)$ as $\varepsilon \to 0^+$, one has that $z(t)$ satisfies equation (R) a.e. on $[0,T]$. By uniqueness, $z(t) = R(t)x$, establishing (1.2). An entirely similar argument with $L^1(0,T;X)$ replaced by $C(0,T;X)$ and assuming (H_5) establishes (1.5).

<u>A an unbounded operator satisfying (H_1).</u> Using the assumptions (H_1), (H_2), (H_4) we define the operator $\tilde{R}$ by the relation

$$\tilde{R}(t)x = \int_0^\infty e^{-\omega A}x \, d\mu_t(\omega) \qquad (x \in X), \tag{2.15}$$

for those $t \in [0,T]$ for which $\mu_t(\omega)$ is defined, and define $\tilde{R}(t)x = 0$ $(x \in X)$ otherwise. Similarly, using assumptions (H_1), (H_2), (H_4), (H_5) we define the operator $\tilde{S}$ by the relation

$$\tilde{S}(t)x = \int_0^\infty e^{-\omega A}x \, d\nu_t(\omega) \qquad (x \in X), \ t \in [0,T] . \tag{2.16}$$

The measures μ_t and ν_t in (2.15) and (2.16) are defined in (2.6) and (2.7) respectively. We point out that the operators $\tilde{R}$ and $\tilde{S}$ will be identified with the operators R and S of Theorem 1 after Lemma 2.5 below. By (H_1) and elementary semigroup theory $\tilde{R}$ and $\tilde{S}$ are bounded operators in X, X_A, and X_{A^2}; we have the estimates:

$$\|\tilde{R}(t)x\| \leq a(t)\|x\| \quad (t \in [0,T];\ x \in X) \tag{2.17}$$

and also

$$\|\tilde{S}(t)x\| \leq \|x\| \quad (t \in [0,T];\ x \in X)\ . \tag{2.18}$$

Define

$$J_\lambda = (I + \lambda A)^{-1} \quad (\lambda \geq 0)$$

and the Yosida approximation A_λ of A by

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda) \quad (\lambda > 0)\ ;$$

recall that by (H_1) J_λ is a contraction on X for every $\lambda \geq 0$ and that, see [19; Cor. 2, p. 241] where the notation is different,

$$A_\lambda x = J_\lambda Ax = AJ_\lambda x \quad (x \in X_A)\ . \tag{2.19}$$

We also need to define the operators $\tilde{R}_\lambda$ and $\tilde{S}_\lambda$ respectively by the relations

$$\tilde{R}_\lambda(t)x = \int_0^\infty e^{-\omega A_\lambda} x\, d\mu_t(\omega) \quad (\lambda > 0)\ , \tag{2.20}$$

for those $t \in [0,T]$ for which $\mu_t(\omega)$ is defined and $\tilde{R}_\lambda(t)x = 0$ $(x \in X)$ *otherwise, and*

$$\tilde{S}_\lambda(t)x = \int_0^\infty e^{-\omega A_\lambda} x\, d\nu_t(\omega) \quad (\lambda > 0,\ t \in [0,T],\ x \in X)\ . \tag{2.21}$$

Since A_λ is a bounded operator for every $\lambda > 0$, it follows from uniqueness in the bounded case and from part (ii) that $\tilde{R}_\lambda(t)x = R_\lambda(t)x$ for $t \in [0,T]$ a.e. and $x \in X$, where $R_\lambda(t)x$ *is the unique solution of equation* (R) with A replaced by A_λ. Similarly, $\tilde{S}_\lambda(t)x = S_\lambda(t)x$ for $t \in [0,T]$ and $x \in X$, where $S_\lambda(t)x$ is the unique solution of equation (S) with A replaced by A_λ. We shall use the following properties of the operators $\tilde{R}, \tilde{R}_\lambda, \tilde{S}, \tilde{S}_\lambda$.

<u>Lemma 2.2</u>. <u>Let</u> (H_1), (H_2), (H_4) <u>be satisfied</u>. <u>Let</u> $\tilde{R}$ <u>and</u> $\tilde{R}_\lambda$ <u>be defined by</u> (2.15) <u>and</u> (2.20) <u>respectively</u>. <u>Then</u>

$$\tilde{R}x \in L^1(0,T;X) \quad (x \in X)\ , \tag{2.22}$$

$$(2.23)\qquad \lim_{\lambda\to 0^+} \|\tilde{R}x - \tilde{R}_\lambda x\|_{L^1(0,T;X)} = 0 \quad (x \in X).$$

<u>Moreover, if</u> $v \in L^1(0,T;X)$, <u>then as a function of</u> s

$$(2.24)\qquad \tilde{R}(t-s)v(s) \in L^1(0,T;X) \quad (t \in [0,T] \text{ a.e.}),$$

$$(2.25)\qquad \int_0^t \tilde{R}(t-s)v(s)ds = \tilde{R} * v(t) \in L^1(0,T;X),$$

$$(2.26)\qquad \lim_{\lambda\to 0^+} \|\tilde{R} * v - \tilde{R}_\lambda * v\|_{L^1(0,T;X)} = 0.$$

<u>Finally, if</u> $v_\lambda \to v$ <u>in</u> $L^1(0,T;X)$ <u>as</u> $\lambda \to 0^+$, <u>then</u>

$$(2.27)\qquad \lim_{\lambda\to 0^+} \|\tilde{R} * v - \tilde{R}_\lambda * v_\lambda\|_{L^1(0,T;X)} = 0.$$

<u>Lemma 2.3</u>. <u>Let</u> (H_1), (H_2), (H_4), (H_5) <u>be satisfied</u>. <u>Let</u> $\tilde{S}$ <u>and</u> $\tilde{S}_\lambda$ <u>be defined by</u> (2.16) <u>and</u> (2.21) <u>respectively</u>. <u>Then properties</u> (2.22) - (2.27) <u>hold with</u> $\tilde{R}$ <u>replaced by</u> $\tilde{S}$ <u>and</u> $\tilde{R}_\lambda$ <u>replaced by</u> $\tilde{S}_\lambda$.

<u>Remark 2.4</u>. In Lemmas 2.2 and 2.3 the space X can be replaced by X_A or X_{A^2} without changing the proof. Also if $v, v_\lambda \in L^p(0,T;X)$, $p \geq 1$, then the properties (2.24) - (2.27) hold in $L^p(0,T;X)$. Moreover, in Lemma 2.3 one can replace $L^1(0,T;X)$ by $C([0,T];X)$ in the formulas corresponding to (2.22), (2.23), (2.25) - (2.27).

We only give the proof of Lemma 2.2.

First (2.22) is immediate from (2.17) by integration. To prove (2.23) we observe from (2.20) that

$$(2.28)\qquad \|\tilde{R}_\lambda(t)x\| \leq a(t)\|x\| \quad (x \in X;\ t \in [0,T]).$$

Next, we show

$$(2.29)\qquad \tilde{R}_\lambda(t)x \to \tilde{R}(t)x \quad (\lambda \to 0^+;\ x \in X;\ t \in [0,T] \text{ a.e.}).$$

By semigroup theory, [19],

$$e^{-\omega A_\lambda}x \to e^{-\omega A}x \quad (\lambda \to 0^+)$$

uniformly in ω on compact subsets of $\mathbb{R}^+$, and so (2.28) holds by Lebesgue's dominated convergence theorem. Thus (2.23) follows from (2.28), (2.29), (H_2), and Lebesgue's dominated convergence theorem. By (2.22) $\tilde{R}(t-s)v(s)$, as a function of (s,t), is measurable for $0 \le s \le t \le T$ with values in X. By (2.28) one has

$$\|\tilde{R}_\lambda(t-s)v(s)\| \le a(t-s)\|v(s)\| , \tag{2.30}$$

where by (H_2) $a(t-s)\|v(s)\| \in L^1(0,T;\mathbb{R}^+)$ for $t \in [0,T]$ a.e. Thus one obtains (2.24) by letting $\lambda \to 0^+$ and by applying Lebesgue's dominated convergence theorem in (2.30). To prove (2.25) and (2.26) we integrate (2.30) obtaining

$$\int_0^T\int_0^t \|\tilde{R}_\lambda(t-s)v(s)\|dsdt \le \int_0^T a(t)dt\,\|v\|_{L^1(0,T;X)} .$$

Therefore, (2.25) follows from Fatou's lemma and (2.26) follows by again applying Lebesgue's theorem. Finally, writing

$$\tilde{R} * v - \tilde{R}_\lambda * v_\lambda = (\tilde{R} * v - \tilde{R}_\lambda * v) + (\tilde{R}_\lambda * v - \tilde{R}_\lambda * v_\lambda) ,$$

and using arguments similar to those employed above one obtains (2.27). This completes the proof of Lemma 2.2.

We next establish the uniqueness of solutions of (1.1) when A is an unbounded operator satisfying (H_1).

Lemma 2.5. *Let (H_1), (H_2), (H_4) be satisfied and let $u \in L^1(0,T;X_A)$ be a strong solution of the equation*

$$u + a * Au = 0 .$$

Then $u = 0$.

Proof of Lemma 2.5. For any $\lambda > 0$ we have from the given equation and from (2.19) that

$$J_\lambda u + a * A_\lambda u = 0 ,$$

or equivalently

$$u + a * A_\lambda u = u - J_\lambda u .$$

By using the fact that A_λ is a bounded operator, together with the representation formula (1.11) where A is replaced by A_λ, f_1 is replaced by $u - J_\lambda u$, and R is replaced by $\tilde{R}_\lambda$, and the uniqueness of solutions of (1.1) in the bounded case, we obtain

$$u = u - J_\lambda u - \tilde{R}_\lambda * A_\lambda(u - J_\lambda u)\,, \tag{2.31}$$

where $\tilde{R}_\lambda$ is defined by (2.20). We wish to show that $u - J_\lambda u$ and $A_\lambda(u - J_\lambda u)$ each tend to zero as $\lambda \to 0^+$ in $L^1(0,T;X)$ for $u \in L^1(0,T;X_A)$. We have

$$\int_0^T \|u - J_\lambda u\|(t)dt = \int_0^T \lambda\|A_\lambda u\|(t)dt \le \lambda \int_0^T \|Au\|(t)dt\,,$$

which tends to zero as $\lambda \to 0^+$. Also

$$\|A_\lambda(u - J_\lambda u)\|(t) = \|A_\lambda \lambda A_\lambda u\|(t) = \|\lambda A_\lambda J_\lambda v\|(t)\,,$$

where $v = Au$; thus

$$\|A_\lambda(u - J_\lambda u)\|(t) = \|J_\lambda v - J_\lambda(J_\lambda v)\|(t) \le \|v - J_\lambda v\|(t)\,.$$

But

$$\|v - J_\lambda v\|(t) \le 2\|v\|(t) = 2\|Au\|(t) \in L^1(0,T)\,;$$

moreover,

$$\|v - J_\lambda v\|(t) \to 0 \quad a.e. \text{ on } [0,T]\,,$$

and therefore, by Lebesgue's theorem, $A_\lambda(u - J_\lambda u) \to 0$ as $\lambda \to 0^+$ in $L^1(0,T;X)$ for $u \in L^1(0,T;X_A)$. Letting $\lambda \to 0^+$ in (2.31) and using the above facts together with (2.27) of Lemma 2.2, we obtain $u = 0$. This completes the proof of Lemma 2.5.

We will complete the proof of Theorems 1 and 2 by first noting that Lemma 2.5 establishes the uniqueness assertions in Theorems 1 and 2. To prove Theorem 1, part (i), we shall prove that $\tilde{R}(t)x$ is a solution of equation (R) for $x \in X_A$. We know that $u_\lambda = \tilde{R}_\lambda(t)x$ defined by (2.20) is the unique solution of the approximating equation associated with (R):

$$u_\lambda + a * A_\lambda u_\lambda = a(t)x \qquad (0 \le t \le T;\ x \in X_A)\,. \tag{2.32}$$

By Lemma 2.2 and Remark 2.4

$$u_\lambda \to u = \tilde{R}x \quad \text{in } L^1(0,T;X_A) \text{ as } \lambda \to 0^+\,, \tag{2.33}$$

where $\tilde{R}$ is defined by (2.15). Thus to pass to the limit in equation (2.32) as $\lambda \to 0^+$, it suffices to show that

$$A_\lambda u_\lambda \to Au \quad \text{in} \quad L^1(0, T; X) \quad \text{as} \quad \lambda \to 0^+ .$$

But $A_\lambda u_\lambda = AJ_\lambda u_\lambda$; thus it suffices to show that $J_\lambda u_\lambda \to u$ in $L^1(0, T; X_A)$ as $\lambda \to 0^+$. This is equivalent to showing that

(2.34) $$\lambda A_\lambda u_\lambda = u_\lambda - J_\lambda u_\lambda \to 0 \quad \text{in} \quad L^1(0, T; X_A) \quad \text{as} \quad \lambda \to 0^+ .$$

But

$$\int_0^T \|A_\lambda u_\lambda(s)\| ds = \int_0^T \|J_\lambda A u_\lambda(s)\| ds \le \int_0^T \|Au_\lambda(s)\| ds \le \|u_\lambda\|_{L^1(0, T; X_A)} \le M ,$$

where $M > 0$ is constant and where the last inequality follows from (2.33). This proves (2.34) and shows that $\tilde{R}(t)x$ is a solution of equation (R) for $x \in X_A$ and for $0 \le t \le T$. By the uniqueness result of Lemma 2.5 we identify $\tilde{R}(t)x$ with $R(t)x$ of Theorem 1 and thereby prove (1.2). The a priori estimates (1.3), (1.4) follow from Lemma 2.2 and Remark 2.4. This completes the proof of Theorem 1 (i).

The proof of Theorem 1 (ii), is similar using the approximating equation associated with (S):

$$u_\lambda + a * A_\lambda u_\lambda = x ,$$

where $u_\lambda = \tilde{S}_\lambda(t)x$ defined by (2.21), and Lemma 2.3. This completes the proof of Theorem 1.

To prove Theorem 2 (i) it is sufficient, by Lemmas 2.2 and 2.5, to show that $u = R * g$ is a strong solution of (1.8). To do this we consider the approximating equation associated with (1.8):

(2.35) $$u_\lambda + a * A_\lambda u_\lambda = a * g \qquad (g \in L^1(0, T; X_A)) .$$

We already know, since A_λ is a bounded operator, that $u_\lambda = \tilde{R}_\lambda * g$ is the unique solution of (2.35), and by Lemma 2.2 and Remark 2.4

$$u_\lambda \to u = R * g \quad \text{in} \quad L^1(0, T; X_A) \quad \text{as} \quad \lambda \to 0^+ .$$

One completes the proof of Theorem 2(i) by letting $\lambda \to 0^+$ in (2.35) and by observing as before that $A_\lambda u_\lambda \to Au$ in $L^1(0,T;X)$ as $\lambda \to 0^+$. The estimate (1.10) follows immediately from its validity for $u_\lambda = \tilde{R}_\lambda * g$, together with Lemma 2.2.

To prove Theorem 2(ii) we consider the approximating equation associated with (1.1):

$$u_\lambda + a * A_\lambda u_\lambda = f. \tag{2.36}$$

We first take $f = f_1$ in (H_6). Since A_λ is bounded

$$u_{1\lambda} = f_1 - \tilde{R}_\lambda * A_\lambda f_1$$

is the unique solution of (2.36) with $f = f_1$. We have that $u_{1\lambda} \in L^1(0,T;X_A)$ and by Lemma 2.2 and Remark 2.4

$$u_{1\lambda} \to u_1 = f_1 - R * Af_1 \quad \text{in} \quad L^1(0,T;X_A) \quad \text{as} \quad \lambda \to 0^+.$$

As above, $A_\lambda u_{1\lambda} \to Au_1$ in $L^1(0,T;X)$ as $\lambda \to 0^+$. Thus letting $\lambda \to 0^+$ in (2.36) and using Lemma 2.5, shows that u_1 given by (1.11) is a strong solution of (1.1).

We next take $f = f_2$ in (H_6), and we obtain (1.12) by a completely analogous argument. The estimate (1.13) follows from formulas (1.11), (1.12), together with the estimates (1.4), (1.6), (1.7). This completes the proof of Theorem 2.

3. A a Nonlinear Operator.

In this section we give a nonlinear analogue of Theorems 3 and 4. Let X be a real Banach space and let $P \subseteq X$ be a closed convex cone. Let $A : D(A) \subseteq X \to 2^X$ be a given, possibly multivalued, m-accretive operator [6; p. 139] satisfying the condition

$$(3.1) \qquad (I + \lambda A)^{-1} P \subseteq P \qquad (\lambda > 0) .$$

Let a satisfy (H_2) and (H_4) and let f satisfy (H_3). Consider the equation

$$(3.2) \qquad u(t) + a * Au(t) \ni f(t) \qquad t \in [0, T] ,$$

where $T > 0$. We say that $u \in L^1(0, T;X)$ is a solution of (3.2) on $[0, T]$ if there exists $w \in L^1(0, T;X)$, where $w(t) \in Au(t)$ a.e., such that $u(t) + a * w(t) = f(t)$ a.e. for $t \in [0, T]$.

Theorem 5. Let (H_2), (H_4) be satisfied. Let f satisfying (H_3) be such that

$$(H_7) \begin{cases} \text{for every } \lambda > 0, \ v, \text{ the unique solution of the linear equation} \\ (3.3) \qquad v(t) + \lambda a * v(t) = f(t) \qquad t \in [0, T] \text{ a.e.}, \\ \text{satisfies } v(t) \in P \text{ a.e. on } [0, T] . \end{cases}$$

For every $\lambda > 0$ let u_λ be the unique solution of the equation

$$(3.4) \qquad u_\lambda(t) + a * A_\lambda u_\lambda(t) = f(t) \qquad t \in [0, T] \text{ a.e.},$$

where A_λ is the Yosida approximation of A. If (3.1) is satisfied, then $u_\lambda(t) \in P$ a.e. on $[0, T]$. Moreover, if u is a solution of equation (3.2) such that $u = \text{weak} \lim_{\lambda \to 0} u_\lambda$ in $L^1(0, T;X)$, then $u(t) \in P$ a.e. on $[0, T]$.

Remark 5.1. Under the assumptions of Theorem 5 it follows from Theorems 3 and 4 with $A = \lambda I$ that if $f(t) = a * g(t)$, $g \in L^1(0, T;X)$, then (H_7) is satisfied if $g(t) \in P$ a.e. on $[0, T]$. If $f(t) = u_0 + a * g(t)$, where $u_0 \in P$ and g is as above, then (H_7) is satisfied provided that (H_5) holds. If $f \in W^{1,1}(0, T;X)$, then (H_7) is satisfied provided that (H_5) holds, and that $f(0) \in P$ and $f'(t) \in P$ a.e. on $[0, T]$.

Remark 5.2. If A is linear and satisfies (H_1), equation (3.2) is (1.1); it was shown in section 2 that the unique solution u_λ of (3.4) converges to u, the unique solution of (1.1), under the assumptions of Theorem 2.

Remark 5.3. If $X = H$ a real Hilbert space and if $A = \partial\varphi$, where $\varphi : H \to (-\infty, \infty]$ is convex, l.s.c. and proper, Barbu [1] and Londen [13] establish the existence and uniqueness of the solution u of equation (3.2) as a limit of solutions u_λ of equation (3.4), so that Theorem 5 can be applied to such a nonlinear equation. A generalization to the case when A is a maximal monotone operator on H is carried out by Gripenberg [11]. It should be noted that in the existence theory of [1], [11], and [13] $a(0) > 0$ and finite is essential, while in Theorem 5 $a(0^+) = +\infty$ is permitted.

Proof of Theorem 5. Consider the equation (3.4) written in the equivalent form

$$u_\lambda + \frac{1}{\lambda} a * u_\lambda = f + \frac{1}{\lambda} a * J_\lambda u_\lambda . \tag{3.5}$$

Define $f_\lambda \in L^1(0, T; X)$ to be the unique solution of (3.3) with λ replaced by $\frac{1}{\lambda}$. By (H_7) $f_\lambda(t) \in P$ a.e. on $[0, T]$. It is easily checked using

$$r(t, \tfrac{1}{\lambda}) + \frac{1}{\lambda}\int_0^t a(t-\sigma) r(\sigma, \tfrac{1}{\lambda}) d\sigma = a(t) \quad \text{and} \quad f_\lambda(t) = f(t) - \frac{1}{\lambda}\int_0^t r(t-\sigma, \tfrac{1}{\lambda}) f(\sigma) d\sigma$$

that equation (3.5) is equivalent to the equation

$$u_\lambda = F_\lambda(u_\lambda) , \tag{3.6}$$

where

$$F_\lambda(z)(t) = \frac{1}{\lambda}\int_0^t r(t-\sigma, \tfrac{1}{\lambda}) J_\lambda(z)(\sigma) d\sigma + f_\lambda(t) . \tag{3.7}$$

Observe that F_λ maps $L^1(0, T; X)$ into itself. We prove that some iterate of F_λ is a strict contraction in $L^1(0, T; X)$. Indeed, from (3.7), (H_2) and the contraction property of J_λ (recall A is m-accretive) one has

$$\|F_\lambda(u)(t) - F_\lambda(v)(t)\| \leq \frac{1}{\lambda}\int_0^t |r(t-s, \tfrac{1}{\lambda})| \, \|u(s) - v(s)\| ds . \tag{3.8}$$

Define $b_\lambda(t) = \frac{1}{\lambda} r(t, \frac{1}{\lambda})$ and $b_\lambda^n(t) = b_\lambda * b_\lambda * \cdots * b_\lambda(t)$, where the convolution is taken n times. Iterating (3.8) n times we obtain

(3.9) $$\|F_\lambda^n(u)(t) - F_\lambda^n(v)(t)\| \le b_\lambda^n * \|u - v\|_{L^1(0,T;X)} .$$

For any fixed λ choose n_λ so large that $\int_0^t b_\lambda^{n_\lambda}(\sigma)d\sigma = K_\lambda < 1$; then integrating (3.9) we obtain

(3.10) $$\|F_\lambda^{n_\lambda}(u) - F_\lambda^{n_\lambda}(v)\|_{L^1(0,T;X)} \le K_\lambda \|u - v\|_{L^1(0,T;X)} .$$

Thus (3.6) (and by the equivalence also (3.4)) has a unique solution $u_\lambda \in L^1(0,T;X)$ given by

$$u_\lambda = \lim_{n\to\infty} F_\lambda^n(u_0), \quad \text{for any } u_0 \in L^1(0,T;X) .$$

In particular if $u_0(t) \in P$ a.e. on $[0,T]$ and if assumptions (H_4) and (H_7) are satisfied, then by (3.1) and (3.7) $F_\lambda(u_0)(t) \in P$ a.e. on $[0,T]$ and the same holds for $F_\lambda^n(u_0)(t)$ for every n. Consequently the unique solution of (3.4) $u_\lambda(t) \in P$ a.e. on $[0,T]$. This completes the proof of Theorem 5.

Remark 5.4. From the proof of Theorem 5 it is clear that Theorem 5 provides an alternative, and in fact simpler, treatment of Theorems 3 and 4 in the linear case. However, in the linear case Theorems 1 and 2 provide explicit representations for the operators R and S and hence more information about the solution. Moreover, the method of proof of Theorem 5 can be used to analyse more general situations. For example, let X be the product of n Banach spaces $X_1, X_2, \ldots, X_n$, and interpret equation (3.2) as a system of n equations with $u(t)$, $f(t) \in X$ for $t \in [0,T]$ and the kernel a being a $n \times n$ matrix satisfying (H_2) componentwise, and such the associated matrix resolvent $r(t,\lambda) \ge 0$ componentwise (analogue of (H_4)). Let P be a closed convex cone in X and let A be a m-accretive operator on X for a suitable norm satisfying (3.1). If f satisfies (H_3) and (H_7), then the conclusions of Theorem 5 hold.

Remark 5.5. The proof of Theorem 5 is in the same spirit as the proof of Theorem 1 of Weis [17] for the equation

$$x(t) = f(t) + \int_0^t a(t-s)g(s,x(s))ds$$

where x, f, g have values in $\mathbb{R}^n$ and a is a $n \times n$ matrix $\in L^1_{loc}(0,\infty)$ and where g has "separated structure" in the sense that $g(t,x) = \mathrm{col}(g_i(t,x_i))$, $i = 1,\ldots,n$, where each g_i is locally Lipschitz with respect to x_i uniformly for t bounded. Weiss gives a condition which corresponds to (H_4) and (H_7) which insures that the solution $x(t) \geq 0$ for as long as it exists.

4. Examples.

Example 1. This example is an application of Theorem 5. Consider the equation

$$u(t,x) + a * (-\nabla^2 u(t,x) + \beta(u(t,x)) \ni f(t,x) , \tag{4.1}$$

$0 < t < \infty$, $x \in \Omega$, a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ with u satisfying Dirichlet boundary conditions on Γ. β is a maximal monotone graph on $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$. For simplicity we assume that the kernel a is completely monotonic on $[0,\infty)$; thus (see Remark 1.3) assumptions (H_2), (H_4), (H_5) are satisfied on $[0,T]$ for every $T > 0$. We assume $f \in W^{1,2}_{loc}(0,\infty;X)$, $X = L^2(\Omega)$. To see that equation (4.1) is a particular case of (3.2) define

$$Au = -\nabla^2 u + \beta(u) \quad \text{with} \quad D(A) = \{u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) : \beta(u) \in L^2(\Omega)\} . \tag{4.2}$$

As is known, see Brézis [4], A is the subdifferential of the convex, l.s.c., proper function $\varphi : L^2(\Omega) \to (-\infty, +\infty]$ defined by

$$\varphi(u) = \begin{cases} \frac{1}{2} \int_\Omega (\text{grad } u)^2 dx + \int_\Omega j(u)dx & \text{if } u \in W^{1,2}_0(\Omega),\ j(u) \in L^1(\Omega) \\ +\infty & \text{otherwise} , \end{cases}$$

where j is the unique, convex, l.s.c., proper function mapping $\mathbb{R}$ into $(-\infty, +\infty]$ such that $j(0) = 0$ and $\beta = \partial j$. Thus A is maximal monotone on the Hilbert space $L^2(\Omega)$ and hence A is m-accretive. Thus (4.1) with the boundary condition $u = 0$ on Γ is a particular case of (3.2). Let $f \in W^{1,2}_{loc}(0,\infty;X)$; in particular, $f \in C[0,\infty;X)$ and $f(0)$ is well defined as an element of $L^2(\Omega)$. We assume that $f(0) \in W^{1,2}_0(\Omega)$ and $\int_\Omega j(f(0))dx < \infty$. These assumptions on f imply that (H_3), (H_6) are satisfied. It is now easily checked that all the assumptions Londen [13; Theorem 1] or Barbu [1, Theorem 1] are satisfied and therefore, (4.1) possesses a unique solution u on $[0,T]$ for every $T > 0$ in the sense of the definition given following equation (3.2) above. Moreover, $u = \lim_{\lambda \to 0^+} u_\lambda$ in $L^1(0,T;X)$ (even in $L^2(0,T;X)$) for every $T > 0$, where u_λ is the unique solution of the approximating equation (3.4). We shall apply Theorem 5 with $P = L^2_+(\Omega)$. It is well known that the operator A defined by (4.2) satisfies condition (3.1). Therefore, if we require that condition (H_7) is satisfied - this will be the case. For example, if $f(0) \in P$ and $f'(t) \in P$ a.e. on $[0,\infty)$ (see Remark 5.1), then the solution $u(t)$ of (4.1) is nonnegative a.e. on $(0,\infty)$.

Example 2. This example is an application of Theorem 4 (iii). Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ. On Ω we consider the linear second order differential operator

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i u) + Cu$$

where $a_{ij}, a_i \in C^1(\bar{\Omega})$, $C \in L^\infty(\Omega)$,

$$C \geq 0,\quad C + \sum_i \frac{\partial a_i}{\partial x_i} \geq 0 \quad \text{a.e.},$$

and for some positive constant α

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{a.e.},\quad \xi \in \mathbb{R}^n.$$

We define $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. It is known (see [5]) that A satisfies (H_1) with $X = L^2(\Omega)$. Consider the equation

$$u(t) + a * Au(t) = u_0, \quad t \in [0,T], \tag{4.3}$$

where $u_0 \in L^2(\Omega)$ and where a satisfies assumptions (H_2), (H_4), (H_5) on $[0,T]$. Equation (4.3) has a unique weak solution u (see Remark 2.3); moreover, if $u_0 \in D(A)$, then the solution u is strong. Let j be a convex l.s.c. proper function: $\mathbb{R} \to [0,\infty]$ with $0 \in \partial j(0)$, and we fix $j(0) = 0$. Define $\varphi : X \to [0,\infty]$ by

$$\varphi(v) = \begin{cases} \int_\Omega j(v)dx & \text{if } j(v) \in L^1(\Omega) \\ +\infty & \text{otherwise}. \end{cases}$$

Then by [5, Lemma 2] we have $(A_\lambda x, y) \geq 0$ for every $[x, y] \in \partial\varphi$ and for all $\lambda > 0$. Moreover, by [3; Theorem 4.4] (1.21) is satisfied. Consequently, by Theorem 4(iii), if $j(u_0) \in L^1(\Omega)$, one has

$$\int_\Omega j(u(t))(x)dx \leq \int_\Omega j(u_0)(x)dx, \qquad t \in [0, T] .$$

In particular, if $j(u) = |u|^p$, $1 \leq p < \infty$, one obtains

$$\|u(t)\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)}, \tag{4.4}$$

if $u_0 \in L^p(\Omega)$. Note that the case $p = \infty$ can be obtained by passing to the limit. Inequality (4.4) can be obtained directly from Theorem 1, inequality (1.6), if one uses the known that A satisfies (H_1) with $X = L^p(\Omega)$, $1 \leq p < \infty$ see [5; Theorem 8 and remarks preceding].

Example 3. This example is an application of the linear theory developed in Theorems 1-4 to a nonlinear problem. Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ. Let $\gamma : \mathbb{R} \to \mathbb{R}$, $\gamma(0) = 0$, γ continuous and nondecreasing. Assume that the nonlinear elliptic equation

$$-\nabla^2 u = \gamma(u), \quad u|_\Gamma = 0 \tag{4.5}$$

has a nontrivial, positive solution $u_\infty \in L^\infty(\Omega)$. Let a satisfy (H_2), (H_4), (H_5), for every $T > 0$ and conside the nonlinear integral equation

$$\begin{cases} u(t) + a * (-\nabla^2 u - \gamma(u))(t) = u_0 & (0 \leq t < \infty) , \\ u_0 \in L^\infty(\Omega), \quad u = 0 \text{ on } \Gamma . \end{cases} \tag{4.6}$$

Let $Au = -\nabla^2 u$ with $D(A) = \{u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)\}$. Let $X = L^2(\Omega)$. Then A satisfies (H_1). If u is a solution of (4.6) in the sense that $g = \gamma(u) \in L^\infty(0, \infty; X)$ and u is a weak solution (in the sense of Remark 2.3) of the equation

$$u(t) + a * Au(t) = u_0 + a * g(t), \quad t \in [0, T] \text{ a.e.}, \quad \forall T > 0 ,$$

then by Theorem 2 it is easily shown that u satisfies the nonlinear functional equation

$$u(t) = F_{u_0}(u)(t) \qquad (0 \leq t < \infty) , \tag{4.7}$$

where

$$F_{u_0}(u)(t) = S(t)u_0 + R * \gamma(u)(t) . \tag{4.8}$$

We prove the following result about solutions of (4.7), (4.8).

Theorem 6. Let (H_2), (H_4), (H_5) be satisfied for every $T > 0$. For every $u_0 \in X$ satisfying $0 \leq u_0 \leq u_\infty$, the equation (4.7), (4.8) has a positive maximal solution $u_M \in L^\infty(0,\infty;X)$ and a positive minimal solution $u_m \in L^\infty(0,\infty;X)$, such that if $u \in L^\infty(0,\infty;X)$ is any solution of (4.7), (4.8), then

$$0 \leq u_m(t) \leq u(t) \leq u_M(t) \leq u_\infty \quad \text{a.e. on } (0,\infty) . \tag{4.9}$$

Remark 6.1. If $u \in L^\infty(0,\infty;X)$ is a solution of (4.7), (4.8), then it is easily checked that u is a solution of (4.6) in the sense defined above. Note that if the solution $u \in L^\infty(0,\infty;X)$ satisfies the estimate (4.9), then $u \in L^\infty(0,\infty;L^\infty(\Omega))$, and thus $\gamma(u) \in L^\infty(0,\infty;X)$, as well as $\gamma(u) \in L^1(0,T;X)$ for every $T > 0$. These observations are needed for the definition of weak solution.

Remark 6.2. Theorem 6 also holds if the requirement γ nondecreasing is replaced $\rho u + \gamma(u)$ nondecreasing for some $\rho > 0$. To see this replace $-\nabla^2 u$ by $-\nabla^2 u + \rho u$ and replace $\gamma(u)$ by $\rho u + \gamma(u)$ in (4.5), (4.6) and apply the above analysis.

Remark 6.3. Comparing equations (4.1) of Example 1 and (4.6) and taking $f(t) \equiv u_0$ in (4.1), $u_0 \in L^2_+(\Omega)$ we note that if β is single valued and continuous, equations (4.1) and (4.6) differ only by the sign of the nonlinearity. For equation (4.1) one has existence and uniqueness of solutions on $(0,\infty)$ for every $u_0 \in L^2_+(\Omega)$. By contrast, for equation (4.6) it is known that if equation (4.5) has $u = 0$ as the only nonnegative solution, then equation (4.6) can have a positive solution only on a finite interval $(0,T)$. For example, if $n = 3$ and $\gamma(u) = u^5$, it follows from [16, Remark 3.27] that if Ω is star shaped, then (4.5) has $u = 0$ as the only nonnegative solution. Taking $a(t) \equiv 1$, applying [10, Theorem 2.6 and Remark 2.7], and assuming that $u_0 \geq 0$ and that

$$\int_\Omega u_0(x)\phi_0(x)dx \geq \lambda_0^{1/4} ,$$

where λ_0 is the smallest eigenvalue and the corresponding unique eigenfunction $\phi_0 > 0$ in Ω:

$$-\nabla^2\phi_0 = \lambda_0\phi_0 \quad \text{in } \Omega, \quad \phi_0\Big|_\Gamma = 0 ,$$

then the unique nonnegative solution u of (4.6) exists only on a finite interval.

Proof of Theorem 6. Let $E = L^\infty(0,\infty;X)$ with the usual ordering (i.e. $u, v \in E$, $u \leq v \iff \tilde{u}(t,x) \leq \tilde{v}(t,x)$ a.e. in $(t,x) \in (0,\infty) \times \Omega$, where $\tilde{u}$ and $\tilde{v}$ are elements of the equivalence classes u and v respectively). In E let I denote the interval $[0, u_\infty]$ in the sense of order in E. It can be shown that I is a complete lattice with respect to this ordering. For every $u_0 \in I$ we define the function $\tilde{F}_{u_0}$ by

$$\tilde{F}_{u_0}(u)(t) = S(t)u_0 + R * \tilde{\gamma}(u)(t)$$

where

$$\tilde{\gamma}(u) = \begin{cases} \gamma(u) & \text{if } \mu \leq \|u_\infty\|_{L^\infty(\Omega)} \\ \|u_\infty\|_{L^\infty(\Omega)} & \text{otherwise} , \end{cases}$$

in place of the function F_{u_0} defined by (4.8). Then $\check{F}_{u_0}$ satisfies

(4.10) $$\tilde{F}_{u_0} : I \to I$$

and

(4.11) $\tilde{F}_{u_0}$ is monotone $(u, v \in I$ and $u \leq v \Rightarrow \tilde{F}_{u_0}(u) \leq \tilde{F}_{u_0}(v))$.

Let $u \in I$. Then, by Theorems 3 and 4, $\tilde{F}_{u_0}(u) \geq 0$. Moreover, by the fact that

$$u_\infty = Su_\infty + R * \gamma(u_\infty), \quad \text{we have}$$

$$\tilde{F}_{u_0}(u) = Su_0 + R * \tilde{\gamma}(u) \leq Su_0 + R * \gamma(u_\infty) = S(u_0 - u_\infty) + u_\infty \leq u_\infty;$$ which proves (4.10).

Clearly, (4.11) is evident from Theorem 3. By [2, Theorem 11, p. 115], the operator $\tilde{F}_{u_0}$ has a least and a greatest fixed point in I, which correspond respectively to the solutions u_m and u_M, since $u_m \leq u_M \leq u_\infty$ and therefore, $\tilde{\gamma}(u_m) = \gamma(u_m)$, $\tilde{\gamma}(u_M) = \gamma(u_M)$, and so $\tilde{F}_{u_0}(u_m) = F_{u_0}(u_m)$, $\tilde{F}_{u_0}(u_M) = F_{u_0}(u_M)$. This completes the proof of Theorem 6.

Appendix 1

An assumption which has been used frequently in the literature concerning the kernel a is

$$(A_1)\quad \begin{cases} a(t) \in C(0,T),\ a(t) > 0,\ t \in (0,T), \text{ and} \\ \dfrac{a(t)}{a(t+\sigma)} \text{ nonincreasing as a function of } t \text{ for each} \\ \qquad \sigma > 0,\ 0 < t + \sigma < T, \end{cases}$$

see Friedman [7], Levin [12], Miller [14]. We shall prove that condition (A_1) is equivalent to the condition

(A_2) $\quad a(t) \in C(0,T),\ a(t) > 0 \quad t \in (0,T)$ and $\log a(t)$ convex on $(0,T)$.

Moreover, we first prove a preliminary result.

Lemma 1. Let assumption (A_2) be satisfied. Then for every $\nu > 0$, there exists a function a_ν satisfying (A_2) and $a_\nu \in C^1[0,T]$, and $a_\nu(t) \uparrow a(t)$ as $\nu \downarrow 0^+$ for $t \in (0,T)$.

Proof. Define $b : \mathbb{R} \to (-\infty, +\infty]$ by

$$b(t) = \begin{cases} \log a(t) & \text{if } t \in (0,T) \\ b(0) = \lim\limits_{t \to 0^+} \log a(t),\ b(T) = \lim\limits_{t \to T^-} \log a(t) \\ +\infty & \text{if } t \notin [0,T]. \end{cases}$$

Observe that $a(t) > 0$ on $(0,T)$ and the definition of convexity of $\log a(t)$ on $(0,T)$ excludes $a(0^+) = 0$ and $a(T^-) = 0$. Thus b is convex, l.s.c. and proper on $\mathbb{R}$. Define b_ν, $\nu > 0$, to be the Yosida approximation of b; then, see [3; Proposition 2.11],

$$b_\nu(t) = \min_{y \in \mathbb{R}} \{\frac{1}{2\nu} |y - t|^2 + b(y)\}, \quad t \in \mathbb{R},$$

and $b_\nu \in C^1(\mathbb{R})$, b_ν' satisfies a Lipschitz condition on $\mathbb{R}$ with constant $\frac{1}{\nu}$; moreover $b_\nu(t) \uparrow b(t)$ as $\nu \downarrow 0^+$, $t \in \mathbb{R}$. Define $a_\nu = e^{b_\nu}$ and the result follows. This completes the proof of Lemma 1. Using Lemma 1 we shall prove

Lemma 2. The conditions (A_1) and (A_2) are equivalent.

Proof. That $(A_1) \Rightarrow (A_2)$ follows from

$$\frac{a(t)}{a(t+\sigma)} \geq \frac{a(t+\tau)}{a(t+\tau+\sigma)} \quad (0 < t < t+\sigma,\ t+\tau < t+\sigma+\tau < T);$$

using $a(t) > 0$ and putting $\sigma = \tau$ we obtain

$$a(t)a(t + 2\tau) \geq a^2(t + \tau) .$$

Thus putting $t_1 = t$, $t_2 = t + 2\tau$ we have

$$\log a\left(\frac{t_1 + t_2}{2}\right) \leq \frac{1}{2} \log a(t_1) + \frac{1}{2} \log a(t_2) .$$

We note that in [12; calculation following Theorem 1.3] it is only shown that $(A_1) \Rightarrow a(t)$ convex, with the additional assumption that a is nonincreasing, which is not used. Of course, $\log a(t)$ convex implies $a(t)$ convex.

To prove that $(A_2) \Rightarrow (A_1)$, it is sufficient by Lemma 1 to prove $(A_2) \Rightarrow (A_1)$ with the additional assumption $a \in C'[0, T]$. Then $\log a(t)$ convex implies

$$\frac{a'(t)}{a(t)} \leq \frac{a'(t + \sigma)}{a(t + \sigma)} \qquad (0 < t < t + \sigma < T) .$$

Using $a(t) > 0$ we then have

$$\frac{d}{dt} \frac{a(t)}{a(t + \sigma)} = \frac{a(t + \sigma)a'(t) - a'(t + \sigma)a(t)}{a^2(t + \sigma)} \leq 0 ,$$

which completes the proof of Lemma 2.

Proof of Proposition 1. By Lemma 2 it is sufficient to prove Proposition 1(i) under assumptions (A_1) and (H_2). If, in addition, $a \in C[0, T]$, Proposition 1(i) follows directly from [14, Theorem 1] with $h = f = a$ and $g(x, t) = x$.

Let a satisfy assumptions (A_1) and (H_2). Consider the functions a_ν of Lemma 1. Then by the above remark, the functions $r_\nu(t, \lambda) \geq 0$, $t \in [0, T]$, for every $\lambda > 0$, $\nu > 0$, where $r_\nu(t, \lambda)$ is the resolvent kernel associated with $\lambda a_\nu(t)$. The functions a_ν converge to a in $L^1(0, T)$ as $\nu \downarrow 0^+$, since $a_\nu(t) \uparrow a(t)$ as $\nu \downarrow 0^+$ and $a \in L^1(0, T)$. Therefore, it is easily checked that the functions $r_\nu(\cdot, \lambda)$ converge to $r(\cdot, \lambda)$ in $L^1(0, T)$, where $r(t, \lambda)$ is the resolvent kernel corresponding to $\lambda a(t)$, and $r(t, \lambda) \geq 0$ on $[0, T]$ a.e. This completes the proof of part (i).

Part (ii) is proved in [8; Lemma 2.5] with $h = \lambda a$ (see also [12; Lemma 1.3] with $f \equiv 1$), where the proof is carried out on $(0, \infty)$; this can be applied by extending $a(t)$ as a constant on $[T, \infty)$. This completes the proof of Proposition 1.

REFERENCES

1. V. Barbu, Nonlinear Volterra equations in a Hilbert space, SIAM J. Math. Anal. 6 (1975), 728-741.
2. G. Birkhoff, Lattice Theory, American Mathematical Society, Vol. XXV, 1973.
3. H. Brézis, Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland, 1973.
4. H. Brézis, Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, Contributions to Nonlinear Functional Analysis, Academic Press, 1971, pp. 101-156.
5. H. Brézis and W. Strauss, Semilinear elliptic equations in L^1, J. Math. Soc. Japan 25 (1973), 565-590.
6. M. G. Crandall, An introduction to evolution governed by accretive operators, Dynamical Systems, Vol. 1, Academic Press Inc., 1976, 131-165.
7. A. Friedman, On integral equations of Volterra type, J. Analyse Math. 11 (1963), 381-413.
8. A. Friedman, Monotonicity of solutions of Volterra integral equations in Banach space, Trans. Amer. Math. Soc. 138 (1969), 129-148.
9. A. Friedman and M. Shinbrot, Volterra integral equations in Banach space, Trans. Amer. Math. Soc. 126 (1967), 131-179.
10. H. Fujita, On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations, Proceedings of Symposia in Pure Mathematics, Vol. XVIII, Part 1, Amer. Math. Soc., 1970, p. 105-113.
11. G. Gripenberg, An existence result for a nonlinear Volterra integral equation in Hilbert space, SIAM J. Math. Anal. (to appear).
12. J. J. Levin, Resolvents and bounds for linear and nonlinear Volterra equations, Trans. Amer. Math. Soc. (to appear).
13. S. O. Londen, On an integral equation in a Hilbert space, SIAM J. Math. Anal. (to appear).

14. R. K. Miller, On Volterra integral equations with nonnegative integrable resolvents, J. Math. Anal. Appl. 22 (1968), 319-340.

15. A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Lecture Notes, U. of Maryland, 1974.

16. P. H. Rabinowitz, Variational methods for nonlinear eigenvalue problems. Eigenvalues of Non-Linear Problems, C.I.M.E., Edizioni Cremonese, Roma, 1974, p. 141-195.

17. D. G. Weis, Asymptotic behaviour of some nonlinear Volterra integral equations, J.M.A.A. 49 (1975), 59-87.

18. D. V. Widder, "The Laplace Transform", Princeton University Press, Princeton, N. J., 1941.

19. K. Yosida, Functional Analysis, Springer Verlag, 1966.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: MRC-TSR-1716 (14)	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	(9) Technical
4. TITLE (and Subtitle): ABSTRACT LINEAR AND NONLINEAR VOLTERRA EQUATIONS PRESERVING POSITIVITY. (6)	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): (10) Ph. Clement and J. A. Nohel	8. CONTRACT OR GRANT NUMBER(s): DAAG29-77-G-0004, DAAG29-75-C-0024, MCS 75-21868
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 1 (Applied Analysis)
11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below.	12. REPORT DATE: (11) February 1977
	13. NUMBER OF PAGES: 36
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): (12) 40p.	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

(15) DAAG29-75-C-0024, DAAG29-77-G-0004

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C.
20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Volterra integral equations, abstract integral equations, unbounded operators, m-accretive operators, semigroups, logarithmic convexity, complete monotonicity, resolvent kernel, fundamental solution, positivity of solutions, Yosida approximation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let X be a real or complex Banach space. We study the Volterra equation

$$\text{(v)} \qquad u(t) + \int_0^t a(t-s)Au(s)ds = f(t) \qquad (0 \le t \le T,\ T > 0),$$

where a is a given kernel, A is a bounded or unbounded linear operator from X to X, and f is a given function with values in X (of particular

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

221 200

LB